Exact stationary state for an asymmetric exclusion process with fully parallel dynamics

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The exact stationary state of an asymmetric exclusion process with fully parallel dynamics is obtained using the matrix product ansatz. We give a simple derivation for the deterministic case by a physical interpretation of the dimension of the matrices. We prove the stationarity via a cancellation mechanism, and by making use of an explicit representation of the matrix algebra we easily find closed expressions for the correlation functions in the general probabilistic case. Asymptotic expressions, obtained by making use of earlier results, allow us to derive the exact phase diagram. [S1063-651X(99)03205-5]

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I. INTRODUCTION

In this paper we describe the exact stationary state of an asymmetric exclusion process (ASEP) with fully parallel dynamics and open boundaries. This is a special case of the Nagel-Schreckenberg model for traffic flow [1,2]. Exact results have been known for some time for several update rules such as random sequential [3,4] and sublattice parallel [5–8]. Results are also known for the case of fully parallel dynamics and cylindrical boundary conditions [2,9]. For fully parallel dynamics and open boundary conditions mean field results have been obtained [10,11]. In a recent preprint, Evans, Rajewsky, and Speer [12] presented an exact solution of this model using a site-oriented matrix product ansatz [13]. Using an explicit representation of the resulting algebra, they calculated the current and density profile via generating function techniques.

We will present a simple and physical derivation of the solution for deterministic bulk dynamics. This solution leads us to a bond-oriented matrix product ansatz resulting in a matrix algebra for stochastic bulk dynamics. Using an explicit representation of this algebra, it is shown that difficult recursion relations can be circumvented, and that integral expressions for the current and density profile can be given almost immediately. The resulting integrals can be calculated resulting in closed expressions similar to those of the random sequential case.

The outline of the paper is as follows. In Sec. II the model is defined, and some notation is fixed. In Sec. III we find and solve a simple recursion relation for the deterministic case. This solution can naturally be recast in the form of a matrix product through an interpretation of the dimension of the matrices, which is done in Sec. IV. Section V is concerned with the formulation of a matrix product ansatz for the general case, solutions of which are presented in Sec. VI. In Sec. VII it is shown that with a diagonalization procedure closed expressions for the density profile and other correlation functions are obtained easily. Finally, in Sec. VIII, the phase diagram is derived using asymptotic expression for the density profile and the current.

II. MODEL

The model is defined on a one-dimensional lattice with L sites. Each site may be occupied with a particle or it may be empty. Configurations on the lattice are written as $\{\tau\}=\{\tau_1,\dots,\tau_L\}$ numbering from left to right, where the presence or absence of a particle at site i is denoted by $\tau_i=1$ or $\tau_i=0$. An ASEP is defined by imposing the following dynamics at each time step $t\rightarrow t+1$ for the particles: If there is a particle at site i and if site $i+1$ is empty, it hops to site $i+1$ with probability p and remains at site i with probability $1-p$. If site $i+1$ is occupied, the particle at site i remains there with probability 1. This dynamics is applied to all particles at the same time, hence the name fully parallel. In the deterministic limit $p=1$ this dynamics is also known as the rule-184 cellular automaton which prescribes how the value of τ_i at time $t+1$ depends on the values of τ_{i-1}, τ_i, and τ_{i+1} at time t. Given the configuration $\{\tau\}$ at time t, the configuration $\{\tau'\}$ at time $t+1$ is given by

$$\tau_i'=\hat{p}_{i-1}\tau_{i-1}(1-\tau_i)+(1-\hat{p}_i)\tau_i(1-\tau_{i+1})+\tau_i\tau_{i+1}\quad(i=2,\dots,L-1),\tag{1}$$

where $\hat{p}_i$ are stochastic Boolean variables with mean $\langle\hat{p}_i\rangle=p$. The lattice is coupled to two reservoirs at the first and last sites. Particles may enter the system from the first reservoir if the first site is empty with rate α and they may leave the system with rate β into the second reservoir at the last site. Thus

$$\tau_1'=\hat{\alpha}(1-\tau_1)+(1-\hat{p}_1)\tau_1(1-\tau_2)+\tau_1\tau_2,\tag{2}$$

$$\tau_L'=\hat{p}_{L-1}\tau_{L-1}(1-\tau_L)+(1-\hat{\beta})\tau_L,\tag{3}$$

where $\hat{\alpha}$ and $\hat{\beta}$ are Boolean variables such that $\langle\hat{\alpha}\rangle=\alpha$ and $\langle\hat{\beta}\rangle=\beta$. Note that the dynamical rules have a manifest particle hole symmetry given by

$$\tau_i\rightarrow 1-\tau_{L-i+1},\qquad \hat{\alpha}\leftrightarrow\hat{\beta}.\tag{4}$$

Let us now associate, to every configuration $\{\tau\}$, a vector $\boldsymbol{\tau}$ defining an orthonormal basis of a Hilbert space. A state $\mathbf{P}(t)$ of the system will be any vector in this Hilbert space, i.e.,

$$\mathbf{P}(t)=\sum_{\{\tau\}} P_t(\tau_1,\ldots,\tau_L)\boldsymbol{\tau}. \tag{5}$$

The time evolution of such a state may be written as

$$\mathbf{P}(t+1)=\mathbf{TP}(t), \tag{6}$$

where $\mathbf{T}$ is the transfer matrix. One would like to know the stationary state of this model, i.e., the eigenstate of the transfer matrix with eigenvalue 1. As this state is time independent, from here on we will suppress the temporal suffix of P.

III. RECURRENCE RELATION

For the deterministic bulk dynamics ($p=1$) the bulk relations (1) simplify. Tilstra and Ernst [11] showed that in this case each configuration that can occur in the stationary state may be spatially divided into three parts: a free flow part, a jammed flow part, and an interface of varying width. The free flow part is defined to be that part of the configuration up to the rightmost 00 pair, and consists of isolated particles only. The jammed flow part starts with the leftmost 11 pair, and contains isolated holes only. The jammed flow and the free flow cannot overlap, but they may be separated by an interface consisting of a sequence of 10 pairs. Using this identification the dynamical rules become very simple. Denote the site of the last 0 of the last 00 pair by f, and the site of the first 1 of the first 11 pair by j; then the rules are given by

$$\begin{aligned}
&\tau_i'=\tau_{i-1} \quad (i=2,\ldots,f),\\
&\tau_i'=\tau_{i+1} \quad (i=j,\ldots,L-1),\\
&\tau_i'=\tau_{i-1}=\tau_{i+1} \quad (i=f+1,\ldots,j-1).
\end{aligned} \tag{7}$$

If there are no 00 pairs in a particular configuration we set $f=1$ if $\tau_1=0$; otherwise $f=0$. Similarly, if there are no 11 pairs, $j=L$ if $\tau_L=1$, otherwise $j=L+1$. In these cases the bulk relations (7) are not valid but one has to use the boundary relations (2) and (3). The equations of motion (6) for deterministic bulk dynamics ($p=1$), for the stationary state, induces the equation

$$\begin{aligned}
P(\tau_1,\ldots,\tau_f,(10)^n,\tau_j,\ldots\tau_L)={}&F(\tau_1|\tau_2)J(\tau_L|\tau_{L-1})\bigg[P(\tau_2,\ldots,\tau_f,(10)^{n+1},\tau_j,\ldots\tau_{L-1})\\
&+\sum_{q=0}^{n} P(\tau_2,\ldots,\tau_f,(10)^q01(10)^{n-q},\tau_j,\ldots\tau_{L-1})\bigg].
\end{aligned} \tag{8}$$

Here, F and J are the transition rates for particles entering and leaving the system. They are given by

$$\begin{aligned}
&F(0|0)=1-\alpha, && J(1|1)=1-\beta,\\
&F(1|0)=\alpha, && J(0|1)=\beta,\\
&F(0|1)=1, && J(1|0)=1,\\
&F(1|1)=0, && J(0|0)=0.
\end{aligned} \tag{9}$$

Iteration of Eq. (8) suggests the following ansatz for the probabilities P;

$$P(\tau_1,\ldots,\tau_f,(10)^n,\tau_j,\ldots,\tau_L)=\frac{1}{Z_L}P_{\rm f}(\tau_1,\ldots,\tau_f)P_{\rm I}(n)P_{\rm j}(\tau_j,\ldots,\tau_L), \tag{10}$$

$$P_{\rm f}(\tau_1,\ldots,\tau_f)=x^f\prod_{i=1}^{f-1} F(\tau_i|\tau_{i+1}), \tag{11}$$

$$P_{\rm j}(\tau_j,\ldots,\tau_L)=y^{L-j+1}\prod_{i=j}^{L-1} J(\tau_{i+1}|\tau_i), \tag{12}$$

where x, y, and $P_{\rm I}(n)$ are to be determined, and Z_L is a normalization. Substituting this ansatz into Eq. (8), we find that $P_{\rm I}(n)$ obeys the recursion relation

$$P_{\rm I}(n)=x^{-1}y^{-1}P_{\rm I}(n+1)+(1-\alpha)\times(1-\beta)\sum_{p=0}^{n} (\alpha x^2)^p(\beta y^2)^{n-p}, \tag{13}$$

where we have set $P_{\rm I}(0)=1$. Explicit consideration of the equations of motion for $P(0,\ldots,0)$ and $P(0,\ldots,01)$ determines the ratio between x and y. A convenient choice that fulfills this relation is $x=\beta$, $y=\alpha$. Recursion relation (13) can be solved easily, and all the sums can be performed to give

$$P_{\rm I}(n)=(\alpha\beta)^n\frac{(1-\alpha)\beta^{n+1}-(1-\beta)\alpha^{n+1}}{\beta-\alpha}. \tag{14}$$

We thus have calculated the complete probability distribution function for the stationary state for deterministic bulk dynamics. The normalization in Eq. (10) is given by

$$Z_L=\frac{(1-\alpha^2)\beta^{L+1}-(1-\beta^2)\alpha^{L+1}}{\beta-\alpha}. \tag{15}$$

All expressions remain valid for $\alpha=\beta$ by taking the appropriate limits.

IV. MATRIX PRODUCT

We now will construct a matrix product representation of the stationary state. This will turn out to be useful for calculational reasons, but it also helps us to find the solution for $p<1$. First the vectors $\boldsymbol{\tau}$ are written as product states,

$$\boldsymbol{\tau}=\bigotimes_{i=1}^{L-1}\mathbf{v}(\tau_i,\tau_{i+1}). \tag{16}$$

Next we will show how the form of the solution obtained in Sec. III hints in the right direction.

The exact form of $P_{\mathrm{I}}(n)$ as given by Eq. (14) suggests another view at the structure of the probability distribution function. This becomes clear by rewriting Eq. (14) as

$$P_{\mathrm{I}}(n)=\sum_{k=0}^{n}P_0(2k)-(\alpha\beta)^{1/2}\sum_{k=1}^{n}P_0(2k-1), \tag{17}$$

where P_0 is given by

$$\begin{aligned}P_0(k)&=(xy)^{-1}P_{\mathrm{f}}(\tau_f,\ldots,\tau_{f+k})\\&\quad\times P_{\mathrm{j}}(\tau_{f+k+1},\ldots,\tau_j),\\&=(\alpha\beta)^n\alpha^{n-k/2}\beta^{k/2}.\end{aligned} \tag{18}$$

P_{f} and P_{j} are defined in Eqs. (11) and (12). Thus Eq. (17) tells us that the weight of the interface of width $2n$ is equal to a sum over the positions of a separator. Configurations to the left of this separator are regarded as belonging to the free flow part, and configurations to the right as belonging to the jammed part. There are different prefactors for even and odd positions of the separator. After Eq. (17) is substituted into Eq. (10), one may even place the separator inside the free flow or the jammed flow, as the resulting additional terms vanish. This suggests that the stationary state may be written as a matrix product,

$$\begin{aligned}\mathbf{P}&=\sum_{\{\tau\}}P(\tau_1,\ldots,\tau_L)\boldsymbol{\tau}\\&=\langle W|\begin{pmatrix}\mathbf{F}&\mathbf{S}\\\mathbf{0}&\mathbf{J}\end{pmatrix}^{L-1}|V\rangle/Z_L,\end{aligned} \tag{19}$$

which is to be understood as a normal matrix multiplication, but where the matrix elements are tensored. The matrices **F** and **J** govern the free and jammed flow, respectively, while **S** is a matrix for the separator. Indeed, we find

$$\begin{aligned}\mathbf{F}&=\begin{pmatrix}xF(0|0)\mathbf{v}(00)&xF(0|1)\mathbf{v}(01)\\xF(1|0)\mathbf{v}(10)&0\end{pmatrix}\\&=\begin{pmatrix}\beta(1-\alpha)\mathbf{v}(00)&\beta\mathbf{v}(01)\\\alpha\beta\mathbf{v}(10)&0\end{pmatrix},\end{aligned} \tag{20}$$

$$\begin{aligned}\mathbf{J}&=\begin{pmatrix}0&yJ(1|0)\mathbf{v}(01)\\yJ(0|1)\mathbf{v}(10)&yJ(1|1)\mathbf{v}(11)\end{pmatrix}\\&=\begin{pmatrix}0&\alpha\mathbf{v}(01)\\\alpha\beta\mathbf{v}(10)&\alpha(1-\beta)\mathbf{v}(11)\end{pmatrix},\end{aligned} \tag{21}$$

$$\mathbf{S}=\begin{pmatrix}0&\alpha\beta\mathbf{v}(01)\\-(\alpha\beta)^2\mathbf{v}(10)&0\end{pmatrix}, \tag{22}$$

$$\langle W|=(1,1,0,\alpha),\quad\langle V|=(\beta,0,1,1). \tag{23}$$

In other words, the matrix operator in Eq. (19) has two types of binary indices: the explicit ones, referring to the type of flow, free or jammed, are contracted; and the internal ones, the occupation numbers τ_i, are tensored.

V. ALGEBRA

Having found the stationary state for deterministic bulk dynamics ($p=1$) and its representation as a matrix product, we now reverse the problem and show that the stationary state may be found by also using the matrix product ansatz (MPA) technique for stochastic ($p\neq1$) dynamics. In the following we will derive a matrix algebra from the MPA for the ASEP, i.e., for arbitrary p. We will show that a finite dimensional representation for this algebra can be found for $p=1$ by using the solution found in Sec. IV.

As usual for the MPA, the matrix product (19) will be written as

$$\mathbf{P}={}_{\tau_1}\langle W|\begin{pmatrix}A\\B\\C\\D\end{pmatrix}^{\otimes(L-1)}|V\rangle_{\tau_L}, \tag{24}$$

where (A,B,C,D) is a vector on the basis $\mathbf{v}(00)$, $\mathbf{v}(01)$, $\mathbf{v}(10)$, $\mathbf{v}(11)$, with matrix valued entries. It is to be understood that each entry of the tensor product is bra-ketted with ${}_{\tau_1}\langle W|$ and $|V\rangle_{\tau_L}$. In Eq. (24), the tensored indices are explicit, while the contracted indices are implicit; this in contrast to Eq. (19). In the following we will show that the transfer matrix for the ASEP with fully parallel update rules may be written as $\mathbf{T}=\mathcal{R}\mathcal{T}^{L-1}\mathcal{L}$, for which the following mechanism ensures that Eq. (24) is a stationary state:

$$\begin{aligned}&{}_{\tau_1}\langle W|\mathcal{L}\begin{pmatrix}A\\B\\C\\D\end{pmatrix}={}_{\tau_1}\langle W|\begin{pmatrix}\hat{A}\\\hat{B}\\\hat{C}\\\hat{D}\\\hat{X}\end{pmatrix},\\&\mathcal{R}\begin{pmatrix}\hat{A}\\\hat{B}\\\hat{C}\\\hat{D}\\\hat{X}\end{pmatrix}|V\rangle_{\tau_L}=\begin{pmatrix}A\\B\\C\\D\end{pmatrix}|V\rangle_{\tau_L},\end{aligned} \tag{25}$$

$$\mathcal{T}\left[\begin{pmatrix}\hat{A}\\\hat{B}\\\hat{C}\\\hat{D}\\\hat{X}\end{pmatrix}\otimes\begin{pmatrix}A\\B\\C\\D\end{pmatrix}\right]=\begin{pmatrix}A\\B\\C\\D\end{pmatrix}\otimes\begin{pmatrix}\hat{A}\\\hat{B}\\\hat{C}\\\hat{D}\\\hat{X}\end{pmatrix}. \tag{26}$$

In order to find $\mathcal{L}$, $\mathcal{R}$, and T we introduce the probability distribution

$$P(\tau_1,\ldots,\tau_{i-1};\sigma_i,\tau_{i+1}\ldots,\tau_L), \tag{27}$$

which corresponds to partial updated sequences with $\tau_1,\ldots,\tau_{i-1}$ at time $t+1$ and $\tau_{i+1},\ldots,\tau_L$ at time t. The variable σ_i can attain three different values, say 0, 1, and 2. The value 0 corresponds to a hole both at time t and $t+1$, the value 1 to a particle at time t, and the value 2 to a particle moving into site i at time $t+1$. The introduction of such a third state $\sigma_i=2$ is necessary to incorporate correctly the fully parallel update rule. These probabilities correspond with the matrix product ansatz in the following way:

$$P(\ldots,\tau_{i-1};\sigma_i,\tau_{i+1},\ldots)={}_{\tau_1}\langle W|\cdots Y(\tau_{i-2},\tau_{i-1})\hat{Y}(\tau_{i-1},\sigma_i)Y(\tau_i,\tau_{i+1})\cdots|V\rangle_{\tau_L}, \tag{28}$$

where $\tau_i=\sigma_i$ mod 2. This also shows why we need the fifth matrix $\hat{X}=\hat{Y}(0,2)$ for the intermediate states in Eqs. (25) and (26). The equations, which define **T**, for the probabilities (27) are explicitly given by

$$\begin{gathered}
P(\ldots,0;0,\ldots)=P(\ldots;0,0,\ldots),\\
P(\ldots,0;1,\ldots)=P(\ldots;0,1,\ldots),\\
P(\ldots,0;2,\ldots)=pP(\ldots;1,0,\ldots),\\
P(\ldots,1;0,\ldots)=(1-p)P(\ldots;1,0,\ldots)+P(\ldots;2,0,\ldots),\\
P(\ldots,1;1,\ldots)=P(\ldots;1,1,\ldots)+P(\ldots;2,1,\ldots),\\
P(\ldots,1;2,\ldots)=0.
\end{gathered} \tag{29}$$

These equations immediately determine the matrix $\mathcal{T}$. In a similar fashion the boundary operators $\mathcal{L}$ and $\mathcal{R}$ can be calculated, and are given by

$$\mathcal{L}=\begin{pmatrix} 1-\alpha & 0 & 0 & 0\\ 0 & 1-\alpha & 0 & 0\\ \alpha & 0 & 1-p & 0\\ 0 & \alpha & 0 & 1\\ 0 & 0 & p & 0\end{pmatrix}, \tag{30}$$

$$\mathcal{R}=\begin{pmatrix} 1 & \beta & 0 & 0 & 0\\ 0 & 1-\beta & 0 & 0 & 1\\ 0 & 0 & 1 & \beta & 0\\ 0 & 0 & 0 & 1-\beta & 0\end{pmatrix}.$$

With these definitions, Eq. (26) and (29) imply the following algebra,

$$\begin{gathered}
A\hat{A}=\hat{A}A,\quad A\hat{B}=\hat{A}B,\quad A\hat{X}=p\hat{B}C,\\
B\hat{C}=(1-p)\hat{B}C+\hat{X}A,\quad B\hat{D}=\hat{B}D+\hat{X}B,\\
C\hat{A}=\hat{C}A,\quad C\hat{B}=\hat{C}B,\quad D\hat{D}=\hat{D}D,\\
C\hat{X}=p\hat{D}C,\quad D\hat{C}=(1-p)\hat{D}C.
\end{gathered} \tag{31}$$

There also are relations related to the product form of $\boldsymbol{\tau}$ which forbids that products like $\mathbf{v}(\tau_{i-1},\tau_i)\otimes\mathbf{v}(1-\tau_i,\tau_{i+1})$ occur in any physical quantity,

$$A\hat{D}=A\hat{C}=B\hat{A}=B\hat{B}=B\hat{X}=C\hat{C}=C\hat{D}=D\hat{A}=D\hat{B}=D\hat{X}=0. \tag{32}$$

The boundary conditions from Eq. (25) with the explicit values of the matrices $\mathcal{L}$ and $\mathcal{R}$ [Eq. (30)] become

$$\begin{gathered}
{}_0\langle W|\hat{A}=(1-\alpha){}_0\langle W|A,\quad A|V\rangle_0=\hat{A}|V\rangle_0+\beta\hat{B}|V\rangle_1,\\
{}_0\langle W|\hat{B}=(1-\alpha){}_0\langle W|B,\quad B|V\rangle_1=(1-\beta)\hat{B}|V\rangle_1+\hat{X}|V\rangle_0,\\
{}_1\langle W|\hat{C}=\alpha{}_0\langle W|A+(1-p){}_1\langle W|C,\quad C|V\rangle_0=\hat{C}|V\rangle_0+\beta\hat{D}|V\rangle_1,\\
{}_1\langle W|\hat{D}=\alpha{}_0\langle W|B+{}_1\langle W|D,\quad D|V\rangle_1=(1-\beta)\hat{D}|V\rangle_1,\\
{}_0\langle W|\hat{X}=p{}_1\langle W|C.
\end{gathered} \tag{33}$$

Any solution to Eqs. (31), (32), and (33) thus automatically is a solution for the stationary state via Eqs. (25) and (26). It is, however, not obvious that the cancellation mechanism of Eqs. (25) and (26) is appropriate for this problem. Indeed, we will see that for the case of $p<1$ we will need a slightly weakened version of Eqs. (31) and (33).

By observation of explicit solutions for small system sizes, we have also inferred the following relations between the matrices:

$$\begin{aligned} DCB &= \alpha\beta((1-p)CB + DD + p\alpha\beta D), \\ BCB &= \alpha\beta(AB + BD + p\alpha\beta B), \\ BCA &= \alpha\beta((1-p)BC + AA + p\alpha\beta A), \\ DCA &= \alpha\beta(1-p)(DC + CA + p\alpha\beta C), \end{aligned} \tag{34}$$

with the following boundary conditions:

$$\begin{aligned} {}_0\langle W|A = p\beta(1-\alpha)\,{}_0\langle W|, &\quad B|V\rangle_1 = p\alpha|V\rangle_0, \\ {}_0\langle W|B = p\beta\,{}_1\langle W|, &\quad D|V\rangle_1 = p\alpha(1-\beta)|V\rangle_1, \\ {}_1\langle W|CA = \alpha\beta({}_0\langle W|A + (1-p)\,{}_1\langle W|C), &\quad DC|V\rangle_0 = \alpha\beta(D|V\rangle_1 + (1-p)C|V\rangle_0), \\ {}_1\langle W|CB = \alpha\beta({}_0\langle W|B + {}_1\langle W|D), &\quad BC|V\rangle_0 = \alpha\beta(A|V\rangle_0 + B|V\rangle_1). \end{aligned} \tag{35}$$

We believe Eqs. (34) and (35) to be true for arbitrary system sizes. It turns out that these relations are particularly convenient to obtain a representation. Using this representation in Eqs. (31) and (33) to find the hatted matrices then gives an easy proof of the stationarity of ansatz (24).

Relations (32) can be easily fulfilled by writing

$$\begin{aligned} A = \mathcal{A}\otimes\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, &\quad B = \mathcal{B}\otimes\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \\ C = \mathcal{C}\otimes\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}, &\quad D = \mathcal{D}\otimes\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}, \end{aligned} \tag{36}$$

and similarly for the hatted matrices

$$\hat{A} = \hat{\mathcal{A}}\otimes\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \quad \hat{B} = \hat{\mathcal{B}}\otimes\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},$$

$$\hat{C} = \hat{\mathcal{C}}\otimes\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}, \quad \hat{D} = \hat{\mathcal{D}}\otimes\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}, \tag{37}$$

$$\hat{X} = \hat{\mathcal{X}}\otimes\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}.$$

The boundary vectors are then written as

$$\begin{aligned} |V\rangle_0 = |\mathcal{V}_0\rangle\otimes\begin{pmatrix}1\\ 0\end{pmatrix}, &\quad |V\rangle_1 = |\mathcal{V}_1\rangle\otimes\begin{pmatrix}0\\ 1\end{pmatrix}, \\ {}_0\langle W| = \langle\mathcal{W}_0|\otimes(1,0), &\quad {}_1\langle W| = \langle\mathcal{W}_1|\otimes(0,1). \end{aligned} \tag{38}$$

VI. REPRESENTATIONS

A. Representation for $1-p=(1-\alpha)(1-\beta)$

Rajewski *et al.* [10] already remarked that the product form (24) is exact for ordinary numbers instead of matrices on the line $1-p=(1-\alpha)(1-\beta)$. Indeed, there exist a one-dimensional representation given by

$$\begin{gathered} \mathcal{A} = \beta(1-\alpha), \quad \mathcal{B} = 1, \quad \mathcal{C} = \alpha\beta, \quad \mathcal{D} = \alpha(1-\beta), \\ \hat{\mathcal{A}} = \beta(1-\alpha)^2, \quad \hat{\mathcal{B}} = 1-\alpha, \quad \hat{\mathcal{C}} = \alpha\beta(1-\alpha), \\ \hat{\mathcal{D}} = \alpha, \quad \hat{\mathcal{X}} = \alpha p \end{gathered} \tag{39}$$

with

$$\langle\mathcal{W}_0| = \beta, \quad \langle\mathcal{W}_1| = 1, \quad |\mathcal{V}_0\rangle = 1, \quad |\mathcal{V}_1\rangle = \alpha. \tag{40}$$

B. Representation for $p=1$

In the case of deterministic dynamics in the bulk, $p=1$, a two-dimensional representation of the subalgebra given by Eq. (31) for general values of α and β can be found. The matrices A, B, C, and D can be easily read off from the solution in Sec. IV, and are given by

$$\begin{aligned} \mathcal{A} = \begin{pmatrix}\beta(1-\alpha) & 0\\ 0 & 0\end{pmatrix}, &\quad \mathcal{B} = \begin{pmatrix}\beta & 1\\ 0 & \alpha\end{pmatrix}, \\ \mathcal{C} = \begin{pmatrix}\alpha\beta & -\alpha\beta\\ 0 & \alpha\beta\end{pmatrix}, &\quad \mathcal{D} = \begin{pmatrix}0 & 0\\ 0 & \alpha(1-\beta)\end{pmatrix}. \end{aligned} \tag{41}$$

The representation for the hatted matrices can then be found using algebra (31) and its boundary conditions (33), and is given by

$$\hat{\mathcal{A}} = \begin{pmatrix}\beta(1-\alpha)^2 & 0\\ 0 & 0\end{pmatrix}, \quad \hat{\mathcal{B}} = \begin{pmatrix}\beta(1-\alpha) & 1-\alpha\\ 0 & 0\end{pmatrix},$$

$$\hat{\mathcal{C}} = \begin{pmatrix}\alpha\beta(1-\alpha) & 0\\ 0 & 0\end{pmatrix}, \quad \hat{\mathcal{D}} = \begin{pmatrix}\alpha\beta & 0\\ 0 & \alpha\end{pmatrix}, \tag{42}$$

$$\hat{\mathcal{X}} = \begin{pmatrix}\alpha\beta & \alpha(1-\beta)\\ 0 & \alpha\end{pmatrix},$$

with

$$\langle \mathcal{W}_0| = (\beta,0), \quad \langle \mathcal{W}_1| = (\beta,1),$$
$$|\mathcal{V}_0\rangle = \begin{pmatrix} 1 \\ \alpha \end{pmatrix}, \quad |\mathcal{V}_1\rangle = \begin{pmatrix} 0 \\ \alpha \end{pmatrix}. \tag{43}$$

C. Infinite-dimensional representation

In Appendix A it is explained that for general values of p there exists a basis $\{e_n, f_n\}$ on which the matrices take the following form:

$$\mathcal{D} = \mathcal{B}(1-p) = \alpha\beta q \begin{pmatrix} q & 1 & 0 & 0 & \cdots \\ 0 & q & 1 & 0 & \\ 0 & 0 & q & 1 & \\ 0 & 0 & 0 & q & \\ \vdots & & & & \ddots \end{pmatrix},$$
$$\mathcal{A} = \mathcal{C} = \alpha\beta q \begin{pmatrix} q & 0 & 0 & 0 & \cdots \\ 1 & q & 0 & 0 & \\ 0 & 1 & q & 0 & \\ 0 & 0 & 1 & q & \\ \vdots & & & & \ddots \end{pmatrix}, \tag{44}$$

where $q = \sqrt{1-p}$. This representation does not lead to divergent sums if

$$\alpha, \beta > 1 - \sqrt{1-p}. \tag{45}$$

A representation that is valid for all values of α and β can also be constructed, but this particular one will be useful for us in the sequel. First of all we would like to diagonalize $E = A + B + C + D$ to facilitate further calculations. To find the eigenvalues and eigenvectors of E, we define the vectors

$$|z\rangle\rangle_0 = \sum_{n=0}^{\infty} z^n e_n, \quad |z\rangle\rangle_1 = \sum_{n=0}^{\infty} z^n f_n. \tag{46}$$

It will also be convenient to define the parameters

$$a = \frac{p-\alpha}{\alpha q}, \quad b = \frac{p-\beta}{\beta q}, \tag{47}$$

so that the boundary vectors may be expressed as (see Appendix A)

$$|V\rangle_0 = \kappa \frac{1-\beta}{1-p} |b\rangle\rangle_0, \quad |V\rangle_1 = \kappa |b\rangle\rangle_1, \tag{48}$$

where κ is defined such that the normalization is given by

$${}_0\langle W|V\rangle_0 = \beta, \quad {}_1\langle W|V\rangle_1 = \alpha. \tag{49}$$

In Appendix B it is shown that there exist vectors $|z;z^{-1}\rangle\rangle_\pm$ that are linear combinations of the vectors defined in Eqs. (46), which have the following properties:

$$E|z;z^{-1}\rangle\rangle_+ = \Lambda_+(z)|z;z^{-1}\rangle\rangle_+ = \alpha\beta q\left(z + z^{-1} + q + q^{-1}\right)|z;z^{-1}\rangle\rangle_+, \tag{50}$$

$$E|z;z^{-1}\rangle\rangle_- = \Lambda_-|z;z^{-1}\rangle\rangle_- = \alpha\beta q(q - q^{-1})|z;z^{-1}\rangle\rangle_-. \tag{51}$$

By writing the boundary vectors as linear combinations of these eigenvectors [see Eqs. (B5) and (B7)], the normalization can be expressed as

$$Z_L = ({}_0\langle W| + {}_1\langle W|)E^{L-1}(|V\rangle_0 + |V\rangle_1) = -\tilde{\kappa} \oint_{|z|=1} \frac{\mathrm{d}z}{4\pi \mathrm{i}z}(\Lambda_+(z) - \Lambda_-)\Lambda_+(z)^{L-1}K(z,a)K(z,b), \tag{52}$$

where

$$K(z,c) = \frac{(z - z^{-1})}{(z-c)(z^{-1}-c)}, \quad c = a,b \tag{53}$$

and

$$\tilde{\kappa} = \frac{1-p-(1-\alpha)(1-\beta)}{\alpha\beta(1-p)} = \frac{1-ab}{p}. \tag{54}$$

Expression (52) can be rewritten using the identities in Appendix C. We then find

$$Z_L = -\alpha\beta q \frac{S_L(a) - S_L(b)}{p(\alpha-\beta)} = \frac{S_L(a) - S_L(b)}{a-b}, \tag{55}$$

where

$$S_L(c) = \frac{c}{p}(R_L(c) - \alpha\beta R_{L-1}(c)(q^2-1)) = \frac{c}{p}(R_L(c) + p\alpha\beta R_{L-1}(c)), \tag{56}$$

$$R_L(c)=(\alpha\beta q)^L\sum_{n=0}^{L}\sum_{m=0}^{n}(q+q^{-1}-2)^{L-n}\binom{L}{n}\binom{2n-m}{n}\frac{m+1}{n+1}(1+c)^m \tag{57}$$

$$\overset{(c<1)}{=}-\oint_{|z|=1}\frac{\mathrm{dz}}{4\pi \mathrm{iz}}\Lambda_+(z)^L K(z,c)(z-z^{-1}). \tag{58}$$

Note that the integral representation (58) for $R_L(c)$ is only valid for $c<1$. Under this condition it can be calculated to give Eq. (57). Equation (57), however, is valid for all values of c, which may be checked explicitly for small system sizes or by using another infinite-dimensional representation for $c>1$.

In deriving the phase diagram we will need the large L behavior of $R_L(c)$. Expression (57) for $R_L(c)$ is similar to that of the ASEP with a random sequential update; its asymptotics can be calculated similarly [4,12]. By identifying the terms that have the largest contribution to the sums we find that $n\sim\sigma L$ and $m\sim 2/(1-c)$ for $c<1$, $m\sim\sqrt{2\sigma L}$ for $c=1$, and $m\sim(c-1)\sigma L/c$ for $c>1$, which give

$$R_L(c)\approx\frac{1}{\sqrt{\pi}}\left(\frac{2}{1-c}\right)^2\Lambda_+(1)^L\frac{1}{(\sigma L)^{3/2}}\quad\text{for } c<1 \tag{59}$$

$$\approx\frac{2}{\sqrt{\pi}}\Lambda_+(1)^L\frac{1}{(\sigma L)^{1/2}}\quad\text{for } c=1 \tag{60}$$

$$\approx(1-c^{-2})\Lambda_+(c)^L\quad\text{for } c>1, \tag{61}$$

where

$$\sigma=\frac{4}{2+q+q^{-1}},\quad \Lambda_+(1)=\alpha\beta(1+\sqrt{1-p})^2,$$

$$\Lambda_+(a)=\alpha\beta\frac{p^2(1-\alpha)}{\alpha(p-\alpha)}. \tag{62}$$

$\Lambda_+(b)$ is obtained from $\Lambda_+(a)$ by interchanging α and β.

VII. EXPRESSIONS FOR THE CURRENT AND DENSITY

Using the algebra, it is easy to derive the following expression for the current $J_L=p\langle\tau_i(1-\tau_{i+1})\rangle_L$:

$$J_L=p\frac{1}{Z_L}({}_0\langle W|+{}_1\langle W|)E^iCE^{L-i-2}(|V\rangle_0+|V\rangle_1) \tag{63}$$

$$=p\alpha\beta\left(\frac{Z_{L-1}}{Z_L}(1-2J_{L-1})+p\alpha\beta\frac{Z_{L-2}}{Z_L}(1-J_{L-2})\right), \tag{64}$$

from which we find by induction,

$$J_L=p\alpha\beta\frac{Z_{L-1}}{Z_L}(1-J_{L-1}). \tag{65}$$

The density profile is much harder to find from the algebra and our strategy will be to express all correlation function in terms of the eigenvectors of E. In doing so, the correlation functions are easily expressed as integrals over the unit circle, and can be calculated exactly by the residue theorem or asymptotically via the saddle-point method. We first demonstrate this for the current. Calculating the action of C on $|z;z^{-1}\rangle\rangle_+$ using Eq. (B1) and reexpressing it in the eigenvectors $|z;z^{-1}\rangle\rangle_\pm$ we find

$$C|z;z^{-1}\rangle\rangle_+=\alpha\beta q[(1+zq)|z\rangle\rangle_1-(1+z^{-1}q)z^{-1}\rangle\rangle_1]=\alpha\beta\frac{\Lambda_+(z)}{\Lambda_+(z)-\Lambda_-}(|z;z^{-1}\rangle\rangle_+-|z;z^{-1}\rangle\rangle_-). \tag{66}$$

Inserting this into Eq. (63), we find that

$$J_L=-\frac{p\alpha\beta\tilde{\kappa}}{Z_L}\oint_{|z|=1}\frac{\mathrm{dz}}{4\pi\mathrm{iz}}\Lambda_+(z)^{L-1}K(z,a)K(z,b)=\frac{\alpha\beta}{(a-b)Z_L}(aR_{L-1}(a)-bR_{L-1}(b)), \tag{67}$$

which indeed fulfills Eq. (65).

In order to find the density profile, we now calculate the two-point correlator $\langle\tau_i\tau_{i+1}\rangle_L$, which is given by

$$\langle\tau_i\tau_{i+1}\rangle_L=\frac{1}{Z_L}({}_0\langle W|+{}_1\langle W|)E^{i-1}DE^{L-i-1}(|V\rangle_0+|V\rangle_1). \tag{68}$$

This can be given in an integral representation using

$$D|z;z^{-1}\rangle\rangle_{+}=\alpha\beta(1-p)[z(1+zq)|z\rangle\rangle_{1}-z^{-1}(1+z^{-1}q)|z^{-1}\rangle\rangle_{1}]. \tag{69}$$

Re-expressing Eq. (69) as a linear combination of eigenvectors using Eq. (C4), it then follows that

$$\langle\tau_i\tau_{i+1}\rangle_L=-\frac{\alpha^2\beta^2q\tilde{\kappa}}{Z_L}\sum_{n=0}^{\infty}\oint_{|w|=1}\frac{\mathrm{d}w}{2\pi\mathrm{i}w}\Lambda_{+}(w)^{i-1}K(w,a)(1+w^{-1}q)w^{-n}\oint_{|z|=1}\frac{\mathrm{d}z}{2\pi\mathrm{i}z}\Lambda_{+}(z)^{L-i-1}K(z,b)z(1+zq)z^{n}$$

$$=\frac{\alpha^2\beta^2\tilde{\kappa}q^2}{Z_L}\sum_{m=0}^{L-i-1}R_{L-m-2}(a)R_m(b)+\frac{bq}{p}J_L \tag{70}$$

$$=-\frac{\alpha^2\beta^2\tilde{\kappa}q^2}{Z_L}\sum_{m=0}^{i-2}R_m(a)R_{L-m-2}(b)+1-\frac{aq+q^2+1}{p}J_L. \tag{71}$$

Here we have made use of the fact that the product $\Lambda_{+}(w)^{i-1}\Lambda_{+}(z)^{L-i-1}$ can be written as a sum in two ways. Equation (70) is useful for studying the right boundary, while Eq. (71) is more suited for the left boundary. The density profile is given by

$$\langle\tau_i\rangle_L=\langle\tau_i\tau_{i+1}\rangle_L+J_L/p. \tag{72}$$

The easiest way to analyze the density profile is by looking at its lattice derivative

$$t_L(i)=\langle\tau_{i+1}\rangle_L-\langle\tau_i\rangle_L=-\frac{\alpha^2\beta^2\tilde{\kappa}q^2}{Z_L}R_{i-1}(a)R_{L-i-1}(b). \tag{73}$$

The value of the current and the asymptotic behavior of $t_L(i)$ will determine the phase diagram.

VIII. PHASE DIAGRAM

A. Case $p=1$

The two dimensional representation (41) for this case allow a simple evaluation of the current and density. The current takes two different values corresponding to low (LD) and high density (HD) regions.

1. Low density phase LD

Here $\alpha<\beta$ and the current and density profile are given by

$$J_{-}=\frac{\alpha}{1+\alpha}, \tag{74}$$

$$\langle\tau_i\rangle_L=\frac{\alpha}{1+\alpha}\left(1+\frac{1-\beta}{\beta}\mathrm{e}^{-r/\xi}\right), \tag{75}$$

where $\xi^{-1}=-\ln(\alpha/\beta)$ and $r=L-i$. The density profile is flat except near the right boundary where it falls of exponentially from its maximum value $\langle\tau_L\rangle_L$ to the bulk value.

2. Transition line from LD to HD

On this line $\alpha=\beta$. The current is still given by Eq. (74) but the density profile becomes linear,

$$\langle\tau_i\rangle_L=\frac{\alpha}{1+\alpha}\left(1+\frac{1-\alpha}{\alpha}\frac{i}{L}\right). \tag{76}$$

3. High density phase HD

This phase is characterized by $\alpha>\beta$, and the current and density can be obtained from those in the low density phase by the particle hole symmetry (4). They are given by

$$J_{+}=\frac{\beta}{1+\beta}, \tag{77}$$

$$\langle\tau_i\rangle_L=\frac{1}{1+\beta}(1-(1-\alpha)\mathrm{e}^{-i/\xi}), \tag{78}$$

where $\xi^{-1}=-\ln(\beta/\alpha)$. Thus the density profile is flat except near the left boundary where it increases exponentially from its minimum value $\langle\tau_1\rangle_L$ to its bulk value.

B. General values of p

The current [Eq. (67)] may take three different values depending on the parameters α and β. These values correspond to a low density phase, a high density phase and a so called maximum current phase. The density profile in these phases will be calculated and will give rise to a further discrimination of phases within the low and high density phases (see Fig. 1).

1. Low density phase LD_I

This phase is characterized by the values $a>b>1$ or $\alpha<\beta<1-\sqrt{1-p}$. The current and bulk density ρ in this phase is given by

$$J_{-}=\frac{p\alpha\beta}{\Lambda_{+}(a)+p\alpha\beta}=\frac{\alpha(p-\alpha)}{p-\alpha^2},$$
$$\rho=1-J/\alpha. \tag{79}$$

We find an exponential decay of the density profile with a length scale

$$\xi^{-1}=\xi_a^{-1}-\xi_b^{-1}, \tag{80}$$

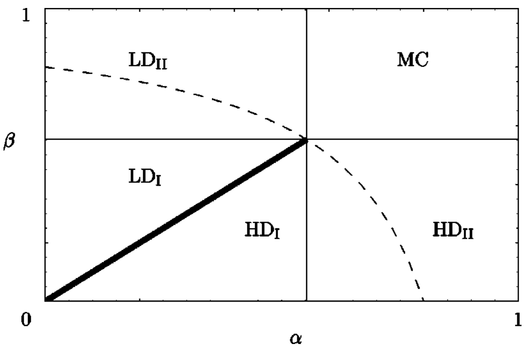

FIG. 1. Phase diagram in the α-β plane. Phase boundaries are at $\alpha,\beta=1-\sqrt{1-p}$, and $\alpha=\beta$, where the transition is discontinuous in the density. On the dashed line, given by $1-p=(1-\alpha)(1-\beta)$, the mean field solution is exact.

with

$$\xi_c^{-1}=-\ln\left(\frac{\Lambda_+(1)}{\Lambda_+(c)}\right), \quad c=a,b. \tag{81}$$

The slope of the density profile is given by (with $r=L-i$)

$$t_L(i)=(b-b^{-1})qJ_-(1-e^{-1/\xi})e^{-(r-1)/\xi}$$
$$=\frac{(1-\alpha)(p-2\beta+\beta^2)}{(p-\alpha^2)(1-\beta)}(1-e^{-1/\xi})e^{-r/\xi}. \tag{82}$$

Since $b>1$ the slope of the density profile is positive and the density approaches its bulk value from above.

2. Transition line from LD_I to LD_{II}

On this line, where $b=1$ or $\beta=1-\sqrt{1-p}$, ξ_b diverges while ξ_a remains finite. The bulk values are the same as in phase $\mathrm{LD_I}$, but the slope of the density profile now becomes

$$t_L(i)=\frac{2qJ_-}{p\sqrt{\sigma\pi}}(1-e^{-1/\xi_a})e^{-(r-1)/\xi_a}r^{-1/2},$$
$$=\frac{\alpha(p-\alpha)}{(p-\alpha^2)}\frac{(1-p)^{1/4}}{\sqrt{\pi}(1-\sqrt{1-p})}$$
$$\times(1-e^{-1/\xi_a})e^{-(r-1)/\xi_a}r^{-1/2}. \tag{83}$$

3. Low density phase LD_{II}

This phase is determined by $b<1<a$ or $\alpha<1-\sqrt{1-p}<\beta$. Here the slope of the density profile still has a power law correction to the exponential, but the power is different.

$$t_L(i)=\frac{qJ_-}{p\sqrt{\pi\sigma}}\frac{\Lambda_+(a)(\Lambda_+(a)-\Lambda_+(b))}{(\Lambda_+(a)-\Lambda_+(1))(\Lambda_+(b)-\Lambda_+(1))}$$
$$\times(1-e^{-1/\xi_a})e^{-r/\xi_a}r^{-3/2}. \tag{84}$$

The slope changes sign at the curve $a=b^{-1}$ or $1-p=(1-\alpha)(1-\beta)$. This is the curve on which the mean field solution is exact, and a one-dimensional representation of our algebra exists. On this line the stationary state completely factorizes and the density profile is flat.

4. Transition line from LD_{II} to the maximal current phase MC

On this transition line the current and bulk value of the density are as in the maximum current phase. The slope of the density profile on this transition line is given by

$$t_L(i)=-\frac{(1-p)^{1/4}}{4\sqrt{\pi}}(i/L)^{-1/2}r^{-3/2}. \tag{85}$$

Near the right boundary the slope decays algebraically as $r^{-3/2}$. Near the left boundary the slope of the density profile decays algebraically with a power of 1/2, but the amplitude is of order $1/L$. Thus up to order $1/L$ corrections the density profile may be regarded as flat near the left boundary.

5. Maximal current phase MC

This part of the phase diagram is characterized by $a,b<1$ or $\alpha,\beta>1-\sqrt{1-p}$. In the maximal current phase the current attains it maximum value which is first reached on its phase boundaries. Its value and the bulk value of the density are given by

$$J_{\max}=\frac{p\alpha\beta}{\Lambda_+(1)+p\alpha\beta}=\frac{1}{2}(1-\sqrt{1-p}), \quad \rho=\frac{1}{2}. \tag{86}$$

The slope of the density profile in this phase is given by

$$t_L(i)=-\frac{(1-p)^{1/4}}{4\sqrt{\pi}}i^{-3/2}(r/L)^{-3/2}. \tag{87}$$

Since its slope is negative the density approaches its bulk value of $\rho=1/2$ from above as $i^{-1/2}$ near the left boundary and from below as $r^{-1/2}$ near the right boundary.

6. High density phases HD_I and HD_{II}

The behavior of the density profile in the high density phases and on their phase boundaries can be obtained from those of the low density phase by the particle hole symmetry [see Eq. (4)],

$$\tau_{i-1}\to 1-\tau_r,$$
$$\alpha\leftrightarrow\beta. \tag{88}$$

7. Coexistence line

This line is characterized by $a=b>1$ or $\alpha=\beta<1-\sqrt{1-p}$. The length $\xi_a=\xi_b$ remains finite but ξ diverges. On this line one finds a linear profile with a positive slope,

$$t_L(i)=\frac{p-2\alpha+\alpha^2}{(p-\alpha^2)L}. \tag{89}$$

In the limit of small rates, i.e., $\alpha=p\tilde{\alpha}$ and $\beta=p\tilde{\beta}$ and $p\to 0$, we recover the results for the ASEP with random sequential update [4,13]. By taking $p\to 1$ the results reduce to those derived in Sec. VIII A. Our results are in perfect agreement with those of Ref. [12].

In all phases and phase boundaries the current and bulk density ρ satisfy the following relation which defines the fundamental diagram

$$J=\tfrac{1}{2}(1-\sqrt{1-4p\rho(1-\rho)}). \tag{90}$$

Following Kolomeisky *et al.* [14], we may understand the phase diagram qualitatively by considering the domain wall dynamics. In this picture two characteristic velocities are important: the domain wall velocity and the collective velocity. The collective velocity is the drift of the center of mass of a momentary local perturbation of the stationary state, and is related to the current by

$$V_{\rm coll}=\frac{\partial}{\partial\rho}J(\rho). \tag{91}$$

This velocity changes sign at $\rho=\frac{1}{2}$, where the current takes its maximum value. For positive domain wall velocity ($\beta>\alpha$) and $\alpha<1-\sqrt{1-p}$, an increase of the left boundary density leads to an increase of the bulk density since $V_{\rm coll}>0$. This happens until the left boundary density equals $\frac{1}{2}$, or $\alpha=1-\sqrt{1-p}$. At this point $V_{\rm coll}$ changes sign, and a perturbation will no longer spread into the bulk. The system is in the maximal current phase and a further increase of the left boundary density does not lead to an increase of the bulk density. For $\beta<\alpha$ the system does not enter the maximal current phase because of the negative domain wall velocity. However, the overfeeding still occurs, which implies that further increase of the left boundary density beyond $\frac{1}{2}$ does not lead to changes of the characteristic length scales in the high density phase. This is seen in the divergence of the length scale ξ_a.

The correlations for the ASEP with fully parallel dynamics are much stronger than for other dynamics. This becomes apparent when considering the relation between the length scales $\xi_{a,b}$ and the curents in the high and low density phases. The lengths $\xi_{a,b}$ can be written as

$$\xi_a^{-1}=\xi_{J_-}^{-1},\quad \xi_b^{-1}=\xi_{J_+}^{-1}, \tag{92}$$

where

$$\xi_J^{-1}=-\ln\left(\frac{J}{1-J}\,\frac{1-J_{\max}}{J_{\max}}\right). \tag{93}$$

This is in contrast to the random sequential and sublattice parallel dynamics [6,7,14], where

$$\xi_J^{-1}=-\ln\left(\frac{J}{J_{\max}}\right). \tag{94}$$

In the latter cases this relation could be obtained directly by considering the domain wall as a biased random walker. We have no simple argument for the fluctuations in the domain wall position that leads to Eq. (93) in the case of fully parallel dynamics.

IX. CONCLUSION

We have presented a stationary state solution of an asymmetric simple exclusion process with fully parallel dynamics. In the case of deterministic bulk dynamics the solution, obtained directly from the master equations, has the form of a product over two-dimensional matrices. In contrast to ASEP's with other dynamics, the matrices depend on two sites instead of one. In the general case the stationary state can still be written as a product over matrices, but of infinite size. The stationarity of this product state can be proven by means of a cancellation mechanism which is a bit weaker than in other cases. We have calculated the exact phase diagram using an explicit representation of the matrix algebra. In this way we could, via a diagonalization procedure, derive expression for the current and the density profile with relative ease.

The results are independent of, and agree with, those of Ref. [12], which were obtained by means of a different ansatz. They prove the strength and the flexibility of the matrix product ansatz, though until recently the fully parallel dynamical models have resisted solution. Even when the resulting algebra is cubic (as in the present paper) or quartic (in Ref. [12]), a representation could be obtained. Of course, now that the formalism has been set up, many other properties of the stationary state can be calculated. Instantaneous correlation functions are relatively straightforward. As our representation includes probability distributions involving consecutive time steps, it is to be expected that the present formalism is capable in principle of producing time dependent correlation functions. A more difficult test of the formalism is the calculation of the distribution of traveling times, for which it is necessary to follow a single particle as it flows through the system.

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APPENDIX A: INFINITE-DIMENSIONAL REPRESENTATION

As an example for finding infinite-dimensional matrices we explicitly construct the representation used in the main text. First we choose the following vectors as a basis:

$$\{e_0,f_0,e_1,f_1,e_2,f_2,\ldots\}, \tag{A1}$$

where

$$\begin{gathered} g_n=(\alpha\beta q)^{-n}(A-\alpha\beta(1-p))^n g_0,\\ e^n=Ag_n,\quad f_n=Cg_n. \end{gathered} \tag{A2}$$

Here $q=\sqrt{1-p}$, and we choose g_0 such that

$$Df_0=\alpha\beta(1-p)f_0,\quad Bf_0=\alpha\beta e_0. \tag{A3}$$

We then find the action of the matrices A, B, C, and D on these vectors from Eqs. (34) and (35). For example, besides Eq. (A3), we find, for $n\geqslant 1$,

$$Df_n=\alpha\beta q(f_{n-1}+qf_n), \tag{A4}$$

$$Bf_n=\alpha\beta q^{-1}(e_{n-1}+qe_n), \tag{A5}$$

so that $\mathcal{D}$ and $\mathcal{B}$ are indeed given by Eq. (44). On the basis of Eq. (A1), the boundary vectors are given by

$$\frac{1-p}{1-\beta}\langle\mathcal{V}_0|=\langle\mathcal{V}_1|=\kappa(1,b,b^2,b^3,\dots), \tag{A6}$$

$$\langle\mathcal{W}_0|=\kappa(1,a,a^2,a^3,\dots), \tag{A7}$$

$$\langle\mathcal{W}_1|=\frac{1}{p\beta}\langle\mathcal{W}_0|\mathcal{B}=\kappa q^{-1}\left(\frac{\alpha q}{p},1,a,a^2,a^3,\dots\right), \tag{A8}$$

where

$$a=\frac{p-\alpha}{\alpha q},\quad b=\frac{p-\beta}{\beta q}, \tag{A9}$$

and

$$\kappa^2=\frac{p(1-p-(1-\alpha)(1-\beta))}{\alpha(1-\beta)}. \tag{A10}$$

This representation does not lead to divergencies if $a,b<1$ or $\alpha,\beta>1-\sqrt{1-p}$. There are many possibilities in choosing the set of basis vectors. We could for example define g_0 in a different way than was done in Eq. (A3). The representation chosen here has some advantages which are exploited in the main text. It is, however, possible to choose a representation which is valid for all values of a and b (Evans, Rajewsky, and Speer gave an explicit example of such a representation [12]). See Derrida *et al.* [13] for a similar discussion.

It turns out that for this representation we can find hatted matrices satisfying the relations on the first line of Eq. (31), but not those on the second line. It is, however, possible to relax the conditions of Eq. (31) a little in the following way. Every matrix will be premultiplied by another matrix. In particular, C or D will be premultiplied by B or D (or ${}_1\langle W|$ at the boundary). We therefore do not have to satisfy the relations on the second line of Eq. (31) identically, but only up to a term that vanishes when acted on by B, D or ${}_1\langle W|$. Since ${}_1\langle W|-q\rangle\rangle_1=0$ and $B|-q\rangle\rangle_1=D|-q\rangle\rangle_1=0$, this is the case if this term is a matrix of which the columns are multiples of $|-q\rangle\rangle_1$. Thus, instead of the algebra obtained from Eq. (26), we find a solution of the algebra implied by

$$_{\tau_1}\langle W|\mathcal{L}\begin{pmatrix}A\\B\\C\\D\end{pmatrix}={}_{\tau_1}\langle W|\begin{pmatrix}\hat{A}\\\hat{B}\\\hat{C}\\\hat{D}\\\hat{X}\end{pmatrix}, \tag{A11}$$

$$\begin{pmatrix}A\\B\\C\\D\end{pmatrix}\otimes\mathcal{R}\begin{pmatrix}\hat{A}\\\hat{B}\\\hat{C}\\\hat{D}\\\hat{X}\end{pmatrix}|V\rangle_{\tau_L}=\begin{pmatrix}A\\B\\C\\D\end{pmatrix}\otimes\begin{pmatrix}A\\B\\C\\D\end{pmatrix}|V\rangle_{\tau_L},$$

$$\begin{pmatrix}A\\B\\C\\D\end{pmatrix}\otimes\mathcal{T}\left[\begin{pmatrix}\hat{A}\\\hat{B}\\\hat{C}\\\hat{D}\\\hat{X}\end{pmatrix}\otimes\begin{pmatrix}A\\B\\C\\D\end{pmatrix}\right]=\begin{pmatrix}A\\B\\C\\D\end{pmatrix}\otimes\begin{pmatrix}A\\B\\C\\D\end{pmatrix}\otimes\begin{pmatrix}\hat{A}\\\hat{B}\\\hat{C}\\\hat{D}\\\hat{X}\end{pmatrix}, \tag{A12}$$

$$_{\tau_1}\langle W|\mathcal{T}\left[\begin{pmatrix}\hat{A}\\\hat{B}\\\hat{C}\\\hat{D}\\\hat{X}\end{pmatrix}\otimes\begin{pmatrix}A\\B\\C\\D\end{pmatrix}\right]={}_{\tau_1}\langle W|\begin{pmatrix}A\\B\\C\\D\end{pmatrix}\otimes\begin{pmatrix}\hat{A}\\\hat{B}\\\hat{C}\\\hat{D}\\\hat{X}\end{pmatrix}. \tag{A13}$$

A solution to this algebra still automatically gives rise to a stationary state. We then find, in addition to Eq. (44),

$$\hat{\mathcal{A}}=\hat{\mathcal{C}}+p^2\alpha\beta\begin{pmatrix}1&(-q)^{-1}&(-q)^{-2}&(-q)^{-3}&\cdots\\-q&1&(-q)^{-1}&(-q)^{-2}&\\(-q)^2&-q&1&(-q)^{-1}&\\(-q)^3&(-q)^2&-q&1&\\\vdots&&&&\ddots\end{pmatrix}, \tag{A14}$$

$$\hat{\mathcal{C}}=\alpha\beta\begin{pmatrix}1-2p&-p^2(-q)^{-1}&-p^2(-q)^{-2}&-p^2(-q)^3&\cdots\\q&1-2p&-p^2(-q)^{-1}&-p^2(-q)^{-2}&\\0&q&1-2p&-p^2(-q)^{-1}&\\0&0&q&1-2p&\\\vdots&&&&\ddots\end{pmatrix}, \tag{A15}$$

$$\hat{\mathcal{B}}=\hat{\mathcal{D}}-p\alpha\beta\begin{pmatrix}1&0&0&0&\cdots\\-q&0&0&0&\\(-q)^2&0&0&0&\\(-q)^3&0&0&0&\\\vdots&&&&\ddots\end{pmatrix},\quad \hat{\mathcal{D}}=\mathcal{D}+p\alpha\beta, \tag{A16}$$

$$\hat{\mathcal{X}}=p\hat{\mathcal{D}}+p\alpha\beta(1-p)\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ -q & 0 & 0 & 0 & \\ (-q)^2 & 0 & 0 & 0 & \\ (-q)^3 & 0 & 0 & 0 & \\ \vdots & & & & \ddots \end{pmatrix}. \tag{A17}$$

APPENDIX B: EIGENVECTORS

It follows from a direct calculation that the actions of the matrices on the vectors defined by Eq. (46) are given by

$$\begin{aligned} A|z\rangle\rangle_0&=\alpha\beta q((z^{-1}+q)|z\rangle\rangle_0-z^{-1}e_0),\\ B|z\rangle\rangle_1&=\alpha\beta(1+zq^{-1})|z\rangle\rangle_0,\\ C|z\rangle\rangle_0&=\alpha\beta q((z^{-1}+q)|z\rangle\rangle_1-z^{-1}f_0),\\ D|z\rangle\rangle_1&=\alpha\beta q(q+z)|z\rangle\rangle_1. \end{aligned} \tag{B1}$$

Taking linear combinations of different vectors $|z\rangle\rangle_1$ and $|z\rangle\rangle_0$ such that terms with e_0 and f_0 drop out [15], we find the following eigenvectors of E:

$$\begin{aligned} |z;z^{-1}\rangle\rangle_\pm=z|z\rangle\rangle_0-z^{-1}|z^{-1}\rangle\rangle_0+\eta_\pm(z)|z\rangle\rangle_1\\ -\eta_\pm(z^{-1})|z^{-1}\rangle\rangle_1, \end{aligned} \tag{B2}$$

$$\eta_+(z)=zq\frac{1+zq}{z+q}, \quad \eta_-(z)=-q. \tag{B3}$$

The eigenvalues corresponding to these vectors follow easily from Eqs. (B1), and are

$$\begin{aligned} \Lambda_+(z)&=\alpha\beta q(z+z^{-1}+q+q^{-1}),\\ \Lambda_-&=\alpha\beta q(q-q^{-1}). \end{aligned} \tag{B4}$$

From now on we take $|z|=1$. The following relations hold for $a<1$ and $b<1$, or $\alpha,\beta>1-\sqrt{1-p}$:

$$({}_0\langle W|+{}_1\langle W|)|z;z^{-1}\rangle\rangle_+=\frac{\kappa}{p\beta}\frac{(z-z^{-1})(\Lambda_+(z)-\Lambda_-)}{(z-a)(z^{-1}-a)}, \tag{B5}$$

$$({}_0\langle W|+{}_1\langle W|)|z;z^{-1}\rangle\rangle_-=0. \tag{B6}$$

The vectors $|V\rangle_0$ and $|V\rangle_1$ can be expressed in the eigenvectors using Eq. (C1), from which we obtain

$$\begin{aligned} &-\oint_{|z|=1}\frac{dz}{4\pi iz}\frac{(z-z^{-1})}{(z-b)(z^{-1}-b)}|z;z^{-1}\rangle\rangle_+\\ &=\kappa^{-1}\frac{1-p}{1-\beta}(|V\rangle_0+|V\rangle_1-\beta^2\kappa|-q\rangle\rangle_1). \end{aligned} \tag{B7}$$

The third term on the right hand side is a null vector, i.e., ${}_1\langle W|-q\rangle\rangle_1=0$ and $E|-q\rangle\rangle_1=0$, and does not enter the calculations.

APPENDIX C: IDENTITIES

The following identity which is frequently used throughout this paper can be conveniently calculated (or looked up in Ref. [16]) by writing the denominator of the integrand as a sum of two geometric series, and using the residue theorem

$$c^{k-1}=-\oint_{|z|=1}\frac{dz}{4\pi iz}\frac{(z^k-z^{-k})(z-z^{-1})}{(z-c)(z^{-1}-c)}, \quad c<1. \tag{C1}$$

In a similar fashion the following integral can be calculated for $c<1$:

$$-\oint_{|z|=1}\frac{dz}{2\pi iz}(2+z+z^{-1})^Lz^n\frac{z-z^{-1}}{(z-c)(z^{-1}-c)}=\sum_{k=0}^{L-n}\sum_{k=0}^{L+n}\left[\binom{2L}{k}c^{L+n-k-1}-\binom{2L}{k}c^{L-n-k-1}\right], \tag{C2}$$

where $0<n\leq L$. Specializing to $n=1$ and rewriting the terms in the sum, we find that

$$-\oint_{|z|=1}\frac{dz}{4\pi iz}\frac{(z-z^{-1})^2(2+z+z^{-1})^N}{(z-c)(z^{-1}-c)}\overset{(c<1)}{=}\sum_{p=0}^{N}\binom{2N-p}{N}\frac{p+1}{N+1}(1+c)^p. \tag{C3}$$

Another important identity that we use to express vectors in terms of the eigenvectors of E is

$$\oint_{|z|=1}\frac{dz}{4\pi iz}\frac{\alpha\beta q^{-1}}{\Lambda_+(z)-\Lambda_-}(|z;z^{-1}\rangle\rangle_+-|z;z^{-1}\rangle\rangle_-)((1+z^{-1}q)_1\langle\langle z|-(1+zq)_1\langle\langle z^{-1}|)$$
$$=\oint_{|z|=1}\frac{dz}{4\pi iz}\left(\frac{z}{z+q}|z\rangle\rangle_1-\frac{1}{1+zq}|z^{-1}\rangle\rangle_1\right)((1+z^{-1}q)_1\langle\langle z|-(1+zq)_1\langle\langle z^{-1}|).$$
$$=I_1-(1-q^2)|-q\rangle\rangle_1\langle\langle -q^{-1}|. \tag{C4}$$

This again can be simply evaluated using the residue theorem and the fact that $q<1$.

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